MA5360 – Assignment 2
Due Date –NO DUE DATE!

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1. Suppose a_n is a sequence with $0 \leq a_n \leq a_{n+1}$ such that
$$\sum_{n=0}^{\infty} 2^n a_{2^n},$$
converges. Show that $\sum_{n=0}^{\infty} a_n$ converges. Is the converse true?

2. Consider the power series
$$\sum_{n=1}^{\infty} \frac{z^n}{n^2}.$$
What is the radius of convergence of the above series? Let f be the sum of the series. Show that f is continuous on the closure of the disk of convergence.

3. Let $f(z) = \sum c_n z^n$ be a power series that converges in $D(a, R), R > 0$ and $f'(0) = c_1 \neq 0$. Prove the equality
$$|f(z) - f(w)| = |z - w| \left|\sum c_n (z^{n-1} + z^{n-2}w + \cdots + zw^{n-2} + w^{n-1})\right|$$
for all $z, w \in D(0, R)$ and consequently that f is injective on the disc $D(0, r)$ if $0 < r < R$ and the following inequality holds:
$$\sum_{n=2}^{\infty} n|c_n| r^{n-1} < |c_1|.$$

4. Let $f(z) = \sum c_n z^n$ be a power series that converges in $D(0, R), R > 0$ and suppose $0 < r < R$. Show:
$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt = \sum |c_n|^2 r^{2n}.$$

5. Look up the definition of real-analytic functions in $\mathbb{R}^n$. Let $f : U \to \mathbb{C}$ be a complex-analytic function. Are $\text{Re} f$ and $\text{Im} f$ real-analytic?

6. Let $f(z) = \sum c_n z^n$ be a convergent power series on $D := D(0, R)$. Show that for each $0 < r < R$ and $m > 1$, $|f^{(m)}(0)| \leq m! M(r)/r^m$ where $M(r) = \sup\{|f(z)| : |z| = r\}$.

7. Let $S \subset \mathbb{R}$ be a subgroup. Show that either S is cyclic or $\overline{S} = \mathbb{R}$.

8. For $z_1, z_2, \alpha_1, \alpha_2 \in \mathbb{C}$, show that
$$L(z_1, \alpha_1) = L(z_2, \alpha_2)$$
iff $z_1 = z_2$ and $\text{Arg}(z_1) = \text{Arg}(z_2)$. Here
$$L(z, \alpha) = \{z + t\alpha : t \geq 0\}.$$

9. Show that for each $n \in \mathbb{N}$, we can find $x_0 \in \mathbb{R}_+$ such that $\exp(x) > x^n$ if $x > x_0$.

10. Show that for $x > 0$ we have $\log x < x$.